Graph Signal Processing (GSP)
(or how I started seeing graphs everywhere)

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Outline

Graphs Everywhere

GSP basics

Contributions

Discussion
Graphs everywhere ...

Graphs provide a flexible model to represent many datasets:

- Examples in Euclidean domains

(a) Computer graphics
(b) Wireless sensor networks
(c) image - graphs
... and then some

Examples in non-Euclidean settings

(a) Social Networks \(^3\), (b) Finite State Machines (FSM)
Graph Signal Processing

- Given a graph (fixed or learned from data)

- and given signals on the graph (set of scalars associated to vertices)

- define frequency, sampling, transforms, etc

- in order to solve problems such as compression, denoising, interpolation, etc

- Overview papers:
  - [Shuman, Narang, Frossard, Ortega, Vandergheysnt, SPM'2013]
  - [Sandryhaila and Moura 2013]
Examples

- Sensor network
  - Relative positions of sensors (kNN), temperature
  - Does temperature vary smoothly?

- Social network
  - Friendship relationship, age
  - Are friends of similar age?

- Images
  - Pixel positions and similarity, pixel values
  - Discontinuities and smoothness
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Graphs 101

- **Graph**: vertices (nodes) connected via some links (edges)
- **Graph Signal**: set of scalar/vector values defined on the vertices.

Graph $G = (\mathcal{V}, E, w)$

- **Vertex Set** $\mathcal{V} = \{v_1, v_2, \ldots\}$
- **Edge Set** $E = \{(v_1, v_2), (v_1, v_3), \ldots\}$
- **Weighted edges** $w$, sets of weights $a_{ij}$
- **Graph Signal** $x = \{x_1, x_2, \ldots\}$
- **Neighborhood, $h$-hop**
  \[ \mathcal{N}_h(i) = \{ j \in \mathcal{V} : \text{hop\_dist}(i, j) \leq h \} \]
Multiple algebraic representations

- Graph $G = (\mathcal{V}, E, w)$.
- Adjacency $A$, $a_{ij}, a_{ji} =$ weights of links between $i$ and $j$ (could be different if graph is directed.)
- Degree $D = \text{diag}\{d_i\}$, in case of undirected graph.
- Various algebraic representations
  - Normalized adjacency $\frac{1}{\lambda_{\text{max}}} A$
  - Laplacian matrix $L = D - A$.
  - Symmetric normalized Laplacian $L = D^{-1/2}L D^{-1/2}$

Discussion:
1. Undirected graphs easier to work with
2. Some applications require directed graphs
3. Graphs with self loops are useful
Graph Fourier Transform (GFT)

- Different results/insights for different choices of operator

- Laplacian $L = D - A = U \Lambda U'$

- Eigenvectors of $L$ : $U = \{u_k\}_{k=1:N}$

- Eigenvalues of $L$ : $\text{diag} \{ \Lambda \} = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$

- Eigen-pair system $\{(\lambda_k, u_k)\}$ provides Fourier-like interpretation — Graph Fourier Transform (GFT)
Eigenvectors of graph Laplacian

(a) $\lambda = 0.00$  (b) $\lambda = 0.04$  (c) $\lambda = 0.20$
(d) $\lambda = 0.40$  (e) $\lambda = 1.20$  (f) $\lambda = 1.49$

Basic idea: increased variation on the graph, e.g., $f^T L f$, as frequency increases
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Graph Filterbank Designs

- Formulation of critically sampled graph filterbank design problem
- Design filters using spectral techniques [Hammond et al. 2009].
- Orthogonal (not compactly supported) [Narang and O. TSP’12]
- Bi-Orthogonal (compactly supported) [Narang and O. TSP’13]
Reconstructed graph-signals for each channel.
Graph Sampling?

- Measure a few nodes to estimate information throughout the graph
- Reconstruct signal in whole graph

Questions:
- What properties enable recovery? Need to define frequency
- How to sample? No obvious regular sampling
- How to reconstruct? Filtering is needed
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Results: Real Datasets

- **USPS**: handwritten digits
  - $x_i = 16 \times 16$ image
  - number of classes = 10
  - $K$-NN graph with $K = 10$
  - $w_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$

- **ISOLET**: spoken letters
  - $x_i \in \mathbb{R}^{617}$ speech features
  - number of classes = 26
  - $K$-NN graph with $K = 10$
  - $w_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$

- **Newsgroups**: documents
  - $x_i \in \mathbb{R}^{3000}$ tf-idf of words
  - number of classes = 10
  - $K$-NN graph with $K = 10$
  - $w_{ij} = \frac{x_i^\top x_j}{\|x_i\|\|x_j\|}$
Assumptions

- Given data matrix \( X = [x_1, \cdots, x_N] \in \mathbb{R}^{n \times N} \).
- The \( k \)-th row of \( X \) (\( k \)-th variable) is attached to \( k \)-th vertex of the graph.
- Each \( x_i \) is a graph signal in an unknown graph.
  - Sensor network: each vertex is a sensor, signal is a measurement/time series
- Data model: attractive Gaussian Markov Random Field (a-GMRF) \( \Leftrightarrow \) Gaussian with a Generalized Laplacian (GL) for precision matrix.
  - \( Q = P + L \) with \( P \) diagonal (self loop matrix) and \( L \) a combinatorial Laplacian.
  - \( Q = (q_{ij}) \) and \( q_{ij} \leq 0 \) for all \( i \neq j \)
- Graph estimation: Max. Likelihood under aGMRF model.

Use block coordinate descent to solve

\[
\min_{Q \text{ is GL}} - \log \det(Q) + \text{tr}(QS),
\]

with \( S = \frac{1}{N} XX^T \).
Experiment: Texture graph

We consider wood textures from Brodatz dataset, with 0 and with 60 degree rotation. For each texture of Brodatz dataset, take $8 \times 8$ blocks, compute their covariance matrix $S$ and solve GL estimation $\rho = 0$.

Texture graphs using our GL estimation (only off diagonal elements of $Q$). The graphs have $|E| = 130$ and $|E| = 117$ edges respectively.
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- Research Question (Twitter Edition):
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  Systems are large and irregular in space/time
  Sampling and interpolation (sensors)
  Variable topology communication networks
  Data reduction to reduce complexity; multiresolution representations
  Control and optimization of large scale systems
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