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A SPATIOTEMPORAL SIR MODEL FOR MODELING THE SPREAD OF AN INFECTIOUS DISEASE

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Digital Tech for COVID_19
Webinar Series
June 12, 2020

CONSERVATION EQUATIONS FOR THREE SPECIES

-Three species: Susceptible (S), Infected (I), Recovered (R) (includes Perished)
-Important Variables: Number/Area. Key to the main mechanism of infection- proximity

$$\frac{\partial \mathcal{N}_i}{\partial T} + \boldsymbol{\nabla} \cdot (\boldsymbol{q}\mathcal{N}_i) = -\boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}}_i) + \mathcal{R}_i \qquad (i = S, I, R)$$

$-\mathcal{N}_i$ is density (number/area) of species i, $\boldsymbol{q}$ is an advective velocity applied uniformly to all populations,
$-\boldsymbol{\mathcal{D}}_i$ is diffusive (or dispersive) flux of i, and $\mathcal{R}_i$ is reaction rate of species (e.g. that converts populations due to infection).

$$\mathcal{N}_S + \mathcal{N}_I + \mathcal{N}_R = \mathcal{N} \quad \text{and} \quad \boldsymbol{\mathcal{D}}_S + \boldsymbol{\mathcal{D}}_I + \boldsymbol{\mathcal{D}}_R = \boldsymbol{\mathcal{D}}$$

$$\frac{\partial \mathcal{N}}{\partial T} + \boldsymbol{\nabla} \cdot (\boldsymbol{q}\mathcal{N}) = -\boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}})$$

$$\boldsymbol{\mathcal{D}}_i = -D\boldsymbol{\nabla}N_i$$

Mass-action Kinetics: $\mathcal{R}_i = \mathrm{K}\mathcal{N}_S\mathcal{N}_I - \Lambda\mathcal{N}_I$; $\mathcal{R}_s = \mathrm{K}\mathcal{N}_S\mathcal{N}_I$; $\mathcal{R}_U = -\Lambda\mathcal{N}_I$ similar to the SIR model

SOME IMPORTANT NOTES

-Λ is inverse {time}: the intrinsic rate at which on average infected individuals recover or die (could add demographics and health conditions in a fine-grained model).

-K is inverse {time*(number/area)} and includes frequency and contact rate (collisions) between individuals. K will be an increasing function of density, up to a maximum "packing". Could use Maxwell-Boltzmann statistics, or better yet other models, since encounters are not elastic, while they last over finite time. Also, spatial distancing practices suggest zero infection below certain density (e.g. corresponding to 6 ft). Take

$$\mathrm{K} = \begin{cases} 0; & \rho < \rho_0 \\ \mathrm{K}_0 F\left(\frac{\rho-\rho_0}{\rho_1-\rho_0}\right); & \rho_0 < \rho < \rho_1 \end{cases}$$

where $F(x)$ is an increasing function of x, $F(0) = 0$, and one can take $\rho_0 = 0.1\ m^{-2}$ and $\rho_1 = 1\ m^{-2}$.

-Meaningless to providing area-wide averages (e.g. for states or countries) without differentiating on density. Distinguish high-density (e.g. urban, stadiums, schools, retirement homes) from low-density (e.g. farms, rural).

-The above formulation assumes that an infected individual can infect a susceptible one at the same rate. This is certainly not true, as most infected individuals are either quarantined or treated.

-The diffusion coefficient can be evaluated by assuming a random walk. For office work, $D = 10^{-3}\frac{m^2}{s}$, which is about two orders of magnitude larger than molecular diffusion in gases.

DIMENSIONLESS NOTATION

$$\frac{\partial s}{\partial t} + (Pe\boldsymbol{v} - 2C\boldsymbol{\nabla} ln\rho) \cdot \boldsymbol{\nabla} s = C\nabla^2 s - R_0(\rho) si$$

$$\frac{\partial i}{\partial t} + (Pe\boldsymbol{v} - 2C\boldsymbol{\nabla} ln\rho) \cdot \boldsymbol{\nabla} i = C\nabla^2 i + R_0(\rho) si - i$$

$$\frac{\partial \rho}{\partial t} + Pe\boldsymbol{v} \cdot \boldsymbol{\nabla}\rho = C\nabla^2 \rho$$

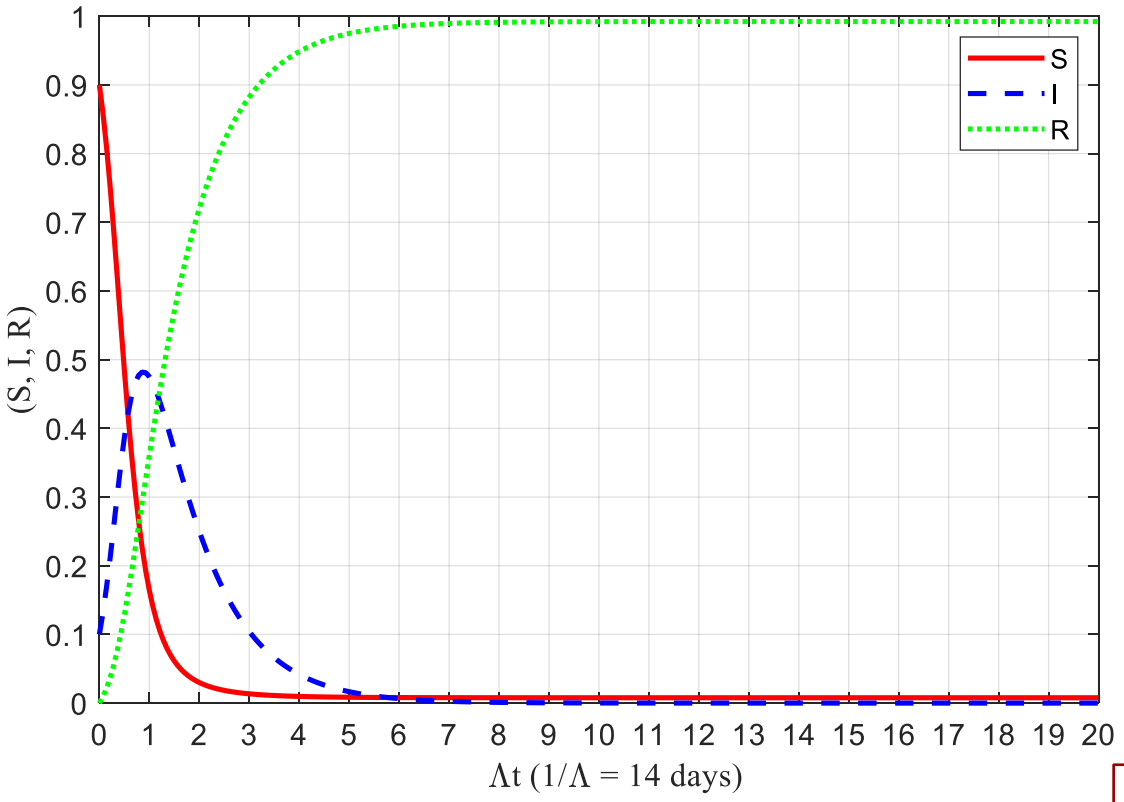
Defined the dimensionless numbers, $Pe = \frac{q}{\Lambda l'}$ which is an equivalent Peclet number, a diffusive number $C = \frac{D}{\Lambda l^2}$, the scaled velocity vector $\boldsymbol{v}$ and $R_0(\rho) = \frac{K}{\Lambda}\rho$.

Equations are hyperbolic (or parabolic) depending on the model taken.
They are also dependent on the spatial density.

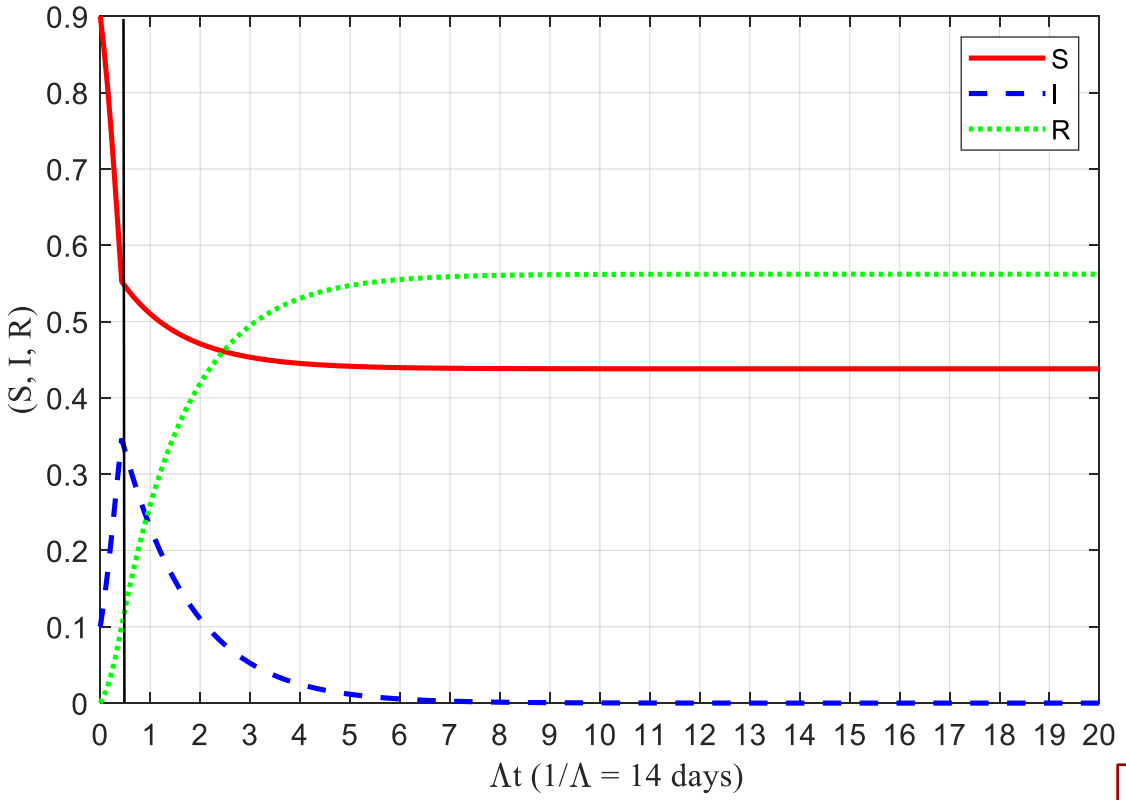
SPECIAL CASES TO SIMULATE

1. No entry or exit in or out of the system, constant K:
This is the standard SIR model (batch reactor model) (the importance of fluctuations, and effects of diffusion) <SI> = <S><I> + $< s'i' >$
2. Enforced health policies after an elapsed period of time τ
(e.g. spatial distancing, lockdown)
3. Entry in the system for a finite time, constant K ("imported infection")
(CSTR model)
4. Spatially variable interactions (effect of diffusion)

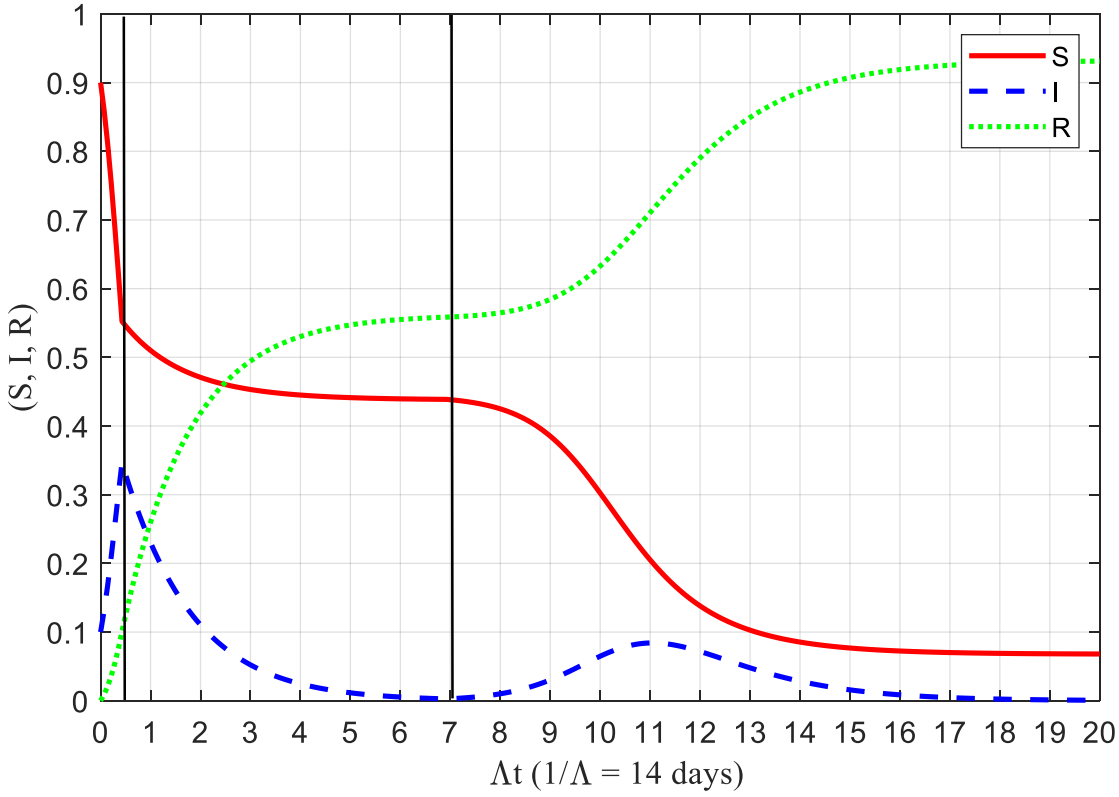
$R_0 = 5$


S
I
R
(S, I, R)
Λt (1/Λ = 14 days)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$R_0 = 5, \tau = 0.5$

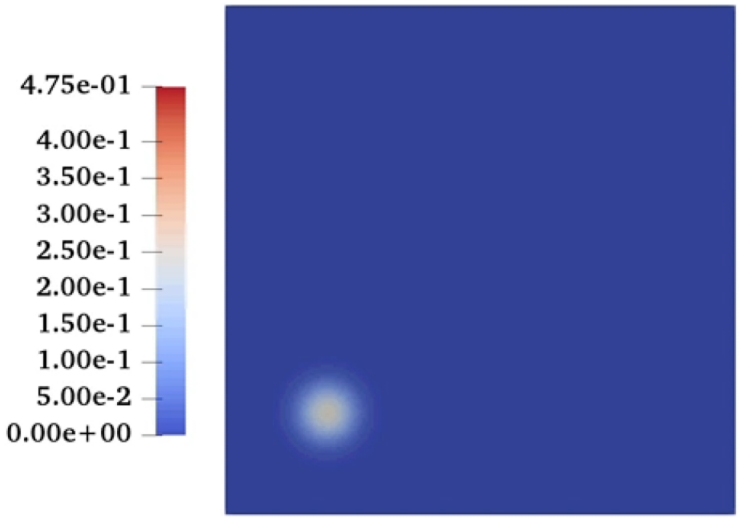

S
I
R
(S, I, R)
Λt (1/Λ = 14 days)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$R_0 = 5,\ \tau = 0.5,\ T = 7$


S
I
R
(S, I, R)
Λt (1/Λ = 14 days)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

R_0 = 5, IC = Gaussian distribution of Infection,

C = 0



C = 3.6e-3