Graph Signal Processing (GSP)

(or how I started seeing graphs everywhere)

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Outline

Graphs Everywhere

GSP basics

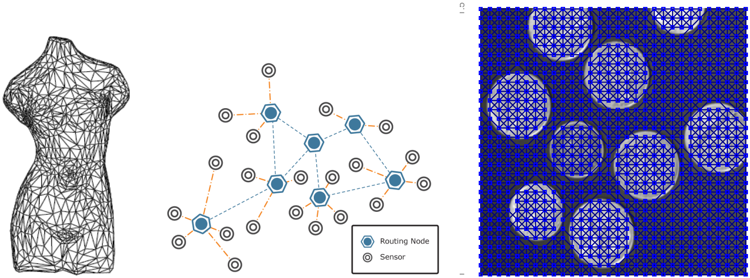
Contributions

Discussion

Graphs everywhere ...

Graphs provide a flexible model to represent many datasets:

- Examples in Euclidean domains

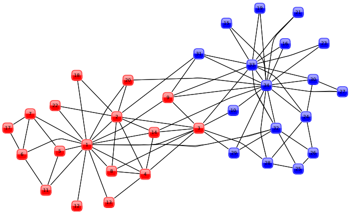


(a) (b) (c)

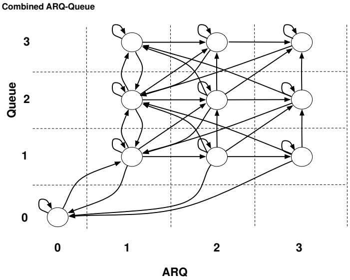
(a) Computer graphics[1] (b) Wireless sensor networks [2] (c) image - graphs

… and then some

- Examples in non-Euclidean settings



(a)



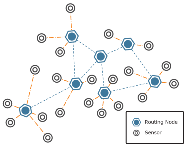
(b)

(a) Social Networks [3], (b) Finite State Machines(FSM)

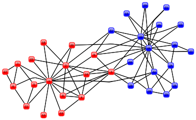
Graph Signal Processing

- Given a graph (fixed or learned from data)
- and given signals on the graph (set of scalars associated to vertices)
- define frequency, sampling, transforms, etc
- in order to solve problems such as compression, denoising, interpolation, etc
- Overview papers:
  [Shuman, Narang, Frossard, Ortega, Vandergheysnt, SPM'2013]
  [Sandryhaila and Moura 2013]

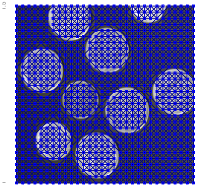
Examples



- Sensor network
  - Relative positions of sensors (kNN), temperature
  - Does temperature vary smoothly?



- Social network
  - Friendship relationship, age
  - Are friends of similar age?

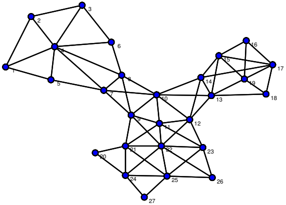


- Images
  - Pixel positions and similarity, pixel values
  - Discontinuities and smoothness

Outline

Graphs 101

- Graph : vertices (nodes) connected via some links (edges)
- Graph Signal: set of scalar/vector values defined on the vertices.



Graph-signal

Graph $G = (\mathcal{V}, E, w)$

Vertex Set $\mathcal{V} = \{v_1, v_2, ...\}$

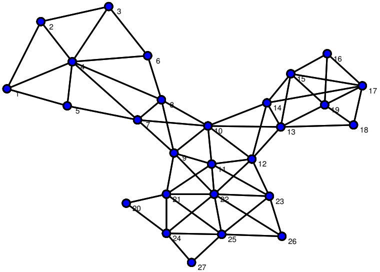
Edge Set $\mathbf{E} = \{(v_1, v_2), (v_1, v_3), ...\}$

Weighted edges w, sets of weights a_{ij}

Graph Signal $\mathbf{x} = \{x_1, x_2, ...\}$

Neighborhood, h-hop
$\mathcal{N}_h(i) = \{j \in \mathcal{V} : hop\_dist(i, j) \leq h\}$

Multiple algebraic representations

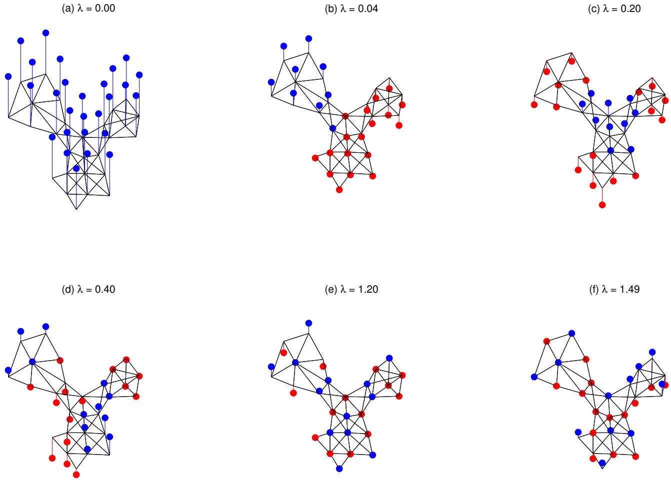


- Graph $G = (\mathcal{V}, E, w)$.
- Adjacency $\mathbf{A}$, a_{ij}, a_{ji} = weights of links between i and j (could be different if graph is directed.)
- Degree $\mathbf{D} = diag\{d_i\}$, in case of undirected graph.
- Various algebraic representations
  - Normalized adjacency $\frac{1}{|\lambda_{max}|}\mathbf{A}$
  - Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$.
  - Symmetric normalized Laplacian $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$
- Discussion:
  1. Undirected graphs easier to work with
  2. Some applications require directed graphs
  3. Graphs with self loops are useful

Graph Fourier Transform (GFT)

- Different results/insights for different choices of operator
- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$
- Eigenvectors of $\mathbf{L}$: $\mathbf{U} = \{\mathbf{u}_k\}_{k=1:N}$
- Eigenvalues of $\mathbf{L}$: $diag\{\boldsymbol{\Lambda}\} = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$
- **Eigen-pair system $\{(\lambda_k, \mathbf{u}_k)\}$ provides Fourier-like interpretation — Graph Fourier Transform (GFT)**

Eigenvectors of graph Laplacian

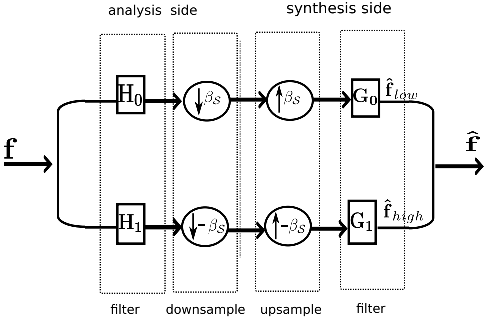


- Basic idea: increased variation **on the graph**, e.g., $\mathbf{f}^t\mathbf{L}\mathbf{f}$, as frequency increases

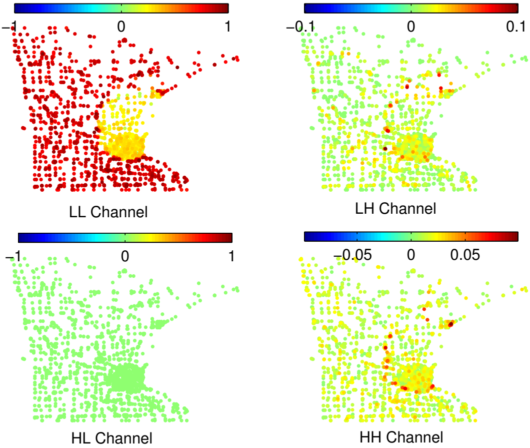
Outline

Graph Filterbank Designs

- Formulation of critically sampled graph filterbank design problem
- Design filters using spectral techniques [Hammond et al. 2009].
- Orthogonal (not compactly supported) [Narang and O. TSP'12]
- Bi-Orthogonal (compactly supported) [Narang and O. TSP'13]



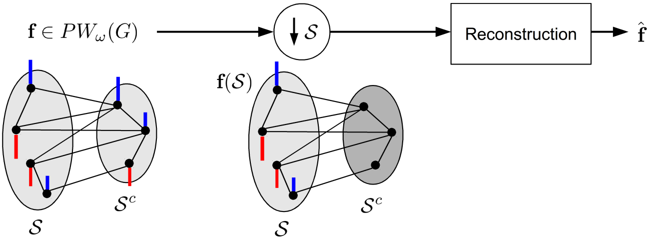
Example



Reconstructed graph-signals for each channel.

Graph Sampling?

- Measure a few nodes to estimate information throughout the graph
- Reconstruct signal in whole graph



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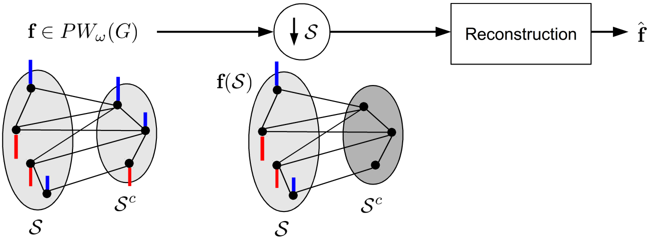
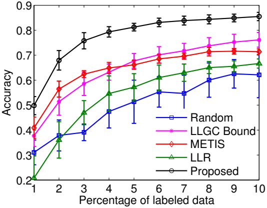


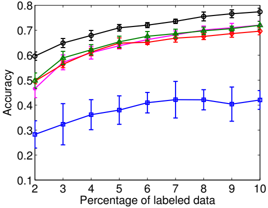
Questions:

- What properties enable recovery? Need to define frequency
- How to sample? No obvious regular sampling
- How to reconstruct? Filtering is needed

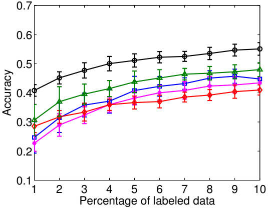
Results: Real Datasets



- USPS: handwritten digits
- $\mathbf{x}_i = 16 \times 16$ image
- number of classes $= 10$
- K-NN graph with $K = 10$
- $w_{ij} = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$



- ISOLET: spoken letters
- $\mathbf{x}_i \in \mathbb{R}^{617}$ speech features.
- number of classes $= 26$
- K-NN graph with $K = 10$
- $w_{ij} = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$



- Newsgroups: documents
- $\mathbf{x}_i \in \mathbb{R}^{3000}$ tf-idf of words
- number of classes $= 10$
- K-NN graph with $K = 10$
- $w_{ij} = \frac{\mathbf{x}_i^\top \mathbf{x}_j}{\|\mathbf{x}_i\| \|\mathbf{x}_j\|}$

Assumptions

- Given data matrix $\mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_N] \in \mathbb{R}^{n \times N}$.
- The k-th row of $\mathbf{X}$ (k-th variable) is attached to k-th vertex of the graph.
- Each $\mathbf{x}_i$ is a graph signal in an unknown graph.
  - Sensor network: each vertex is a sensor, signal is a measurement/time series
- Data model: attractive Gaussian Markov Random Field (a-GMRF) $\Leftrightarrow$ Gaussian with a Generalized Laplacian (GL) for precision matrix.
  - $\mathbf{Q} = \mathbf{P} + \mathbf{L}$ with $\mathbf{P}$ diagonal (self loop matrix) and $\mathbf{L}$ a combinatorial Laplacian.
  - $\mathbf{Q} = (q_{ij})$ and $q_{ij} \leq 0$ for all $i \neq j$
- Graph estimation: Max. Likelihood under aGMRF model.

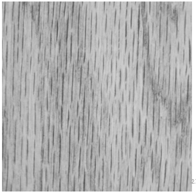
Use block coordinate descent to solve

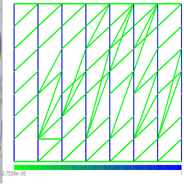
$$\min_{\mathbf{Q} \text{ is GL}} -\log\det(\mathbf{Q}) + \mathrm{tr}(\mathbf{QS}),$$

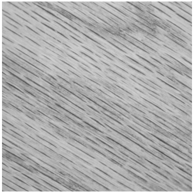
with $\mathbf{S} = \frac{1}{N}\mathbf{X}\mathbf{X}^T$.

Experiment: Texture graph

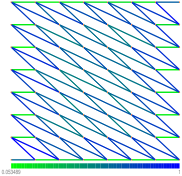
We consider *wood* textures from Brodatz dataset, with 0 and with 60 degree rotation. For each texture of Brodatz dataset, take 8×8 blocks, compute their covariance matrix $\mathbf{S}$ and solve GL estimation $rho = 0$.







wood060



Texture graphs using our GL estimation (only off diagonal elements of $\mathbf{Q}$). The graphs have $|E| = 130$ and $|E| = 117$ edges respectively

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Discussion

- Research Question (Twitter Edition):

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- GSP 4 CPS + IOT @USC-CCI ?

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- Systems are large and irregular in space/time
  - Sampling and interpolation (sensors)

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  - Data reduction to reduce complexity; multiresolution representations

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- Systems are large and irregular in space/time
  - Sampling and interpolation (sensors)
  - Variable topology communication networks
  - Data reduction to reduce complexity; multiresolution representations
  - Control and optimization of large scale systems

References I

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