$\{$Network Flows, Queues, Combinatorial Optimization, ... $\}$
$+$
Dynamics

CCI Mini-Workshop on Theoretical Foundations of CPS
April 14, 2017

USC Viterbi
School of Engineering

Ketan Savla
(ksavla@usc.edu)

University of Southern California

Personal Perspective on Theoretical CPS

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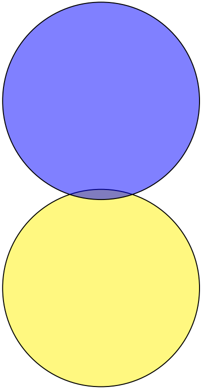
(operations research, network science, decision theory)

Systems Sciences



Physical Sciences

(physics, human behavior, dynamics)

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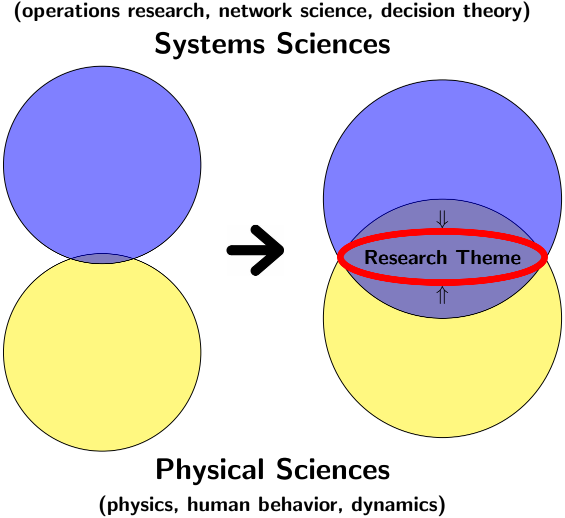

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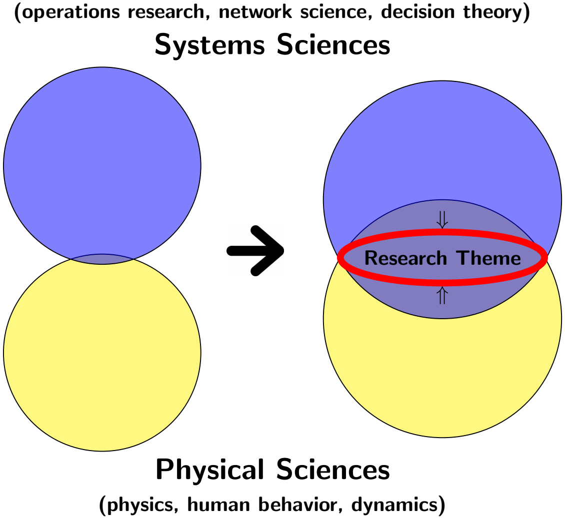

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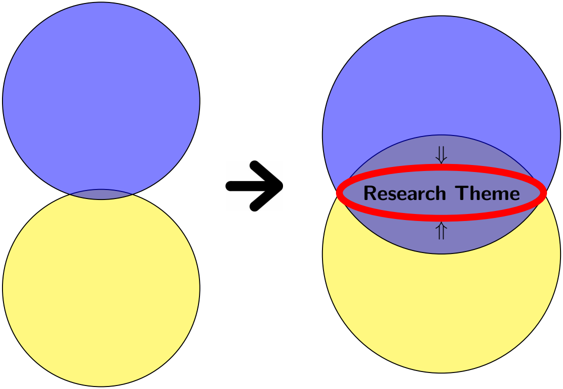

$$\left\{ \begin{array}{c} \text{optimization} \\ \text{queueing systems} \\ \text{network flows} \\ \text{game theory} \end{array} \right\}$$

$$+$$

$$\left\{ \begin{array}{c} \text{motion kinematics} \\ \text{psychology} \\ \text{dynamics} \\ \text{physics} \end{array} \right\}$$

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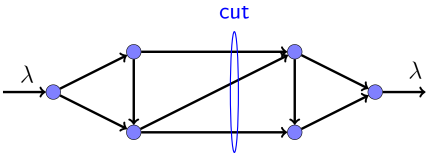
Static Network Flow



- Link flow capacity C_e
- Feasible z:
  - $z_e \leq C_e$
  - $\sum_{\text{incoming}} z_e = \sum_{\text{outgoing}} z_{e'}$

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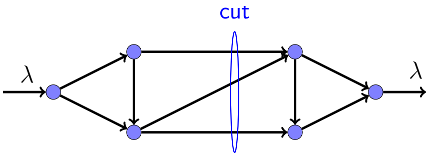

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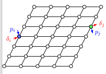
Max flow min cut theorem

$$\lambda \quad \leq \quad \underbrace{\min_{\text{cut}} \sum_{e \in \text{cut}} C_e}_{\text{min cut capacity}} \iff \text{feasible } z$$

Transmission Capacity of the Power Grid



C^{maxtrans}

$$\max_{w \in [w^l, w^u]} \quad \max_{\mu} \quad \mu \qquad \underline{Nonconvex}$$

$$\text{s.t.} \quad -c \leq z(w, \mu) \leq c$$

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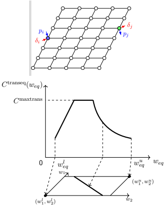

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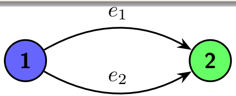

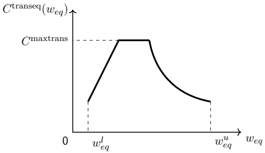

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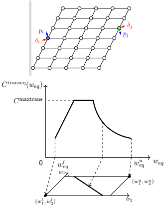

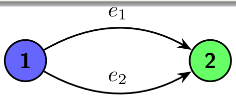

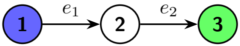

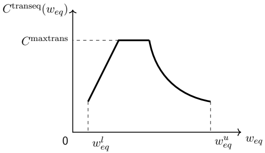

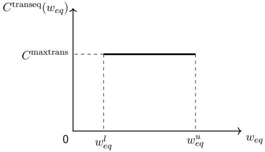

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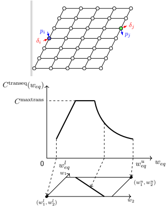
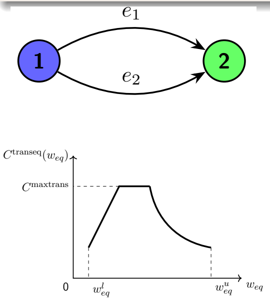
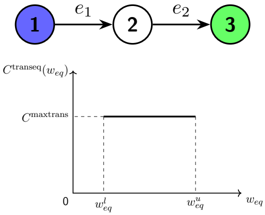
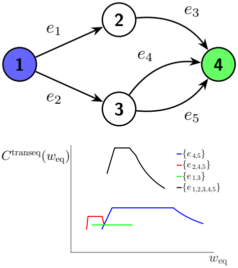




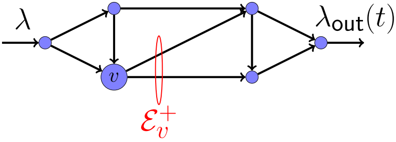




Max Residual Capacity under Local Routing

δ^* =min node residual capacity

$$:= \min_v \sum_{e \in \mathcal{E}_v^+} (C_e - z_e^*)$$



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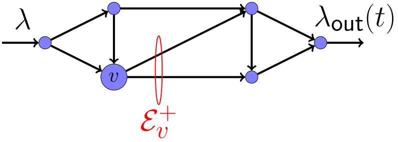
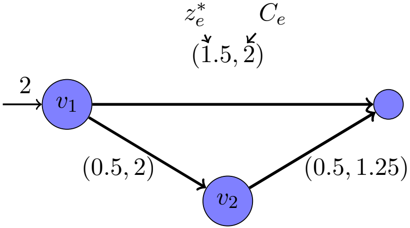


- Min node residual capacity
  $= \min\{1.5 + 1.5, 0.75\}$
  $= 0.75$
- $\delta = 0.6$



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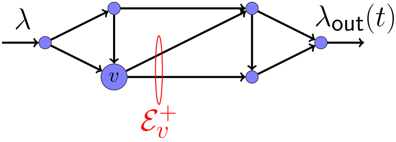

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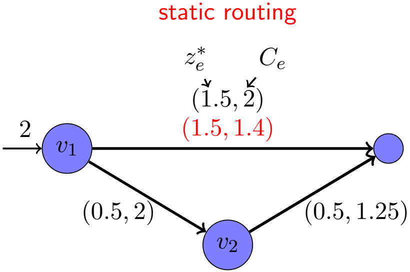

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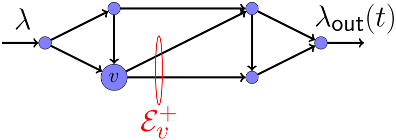

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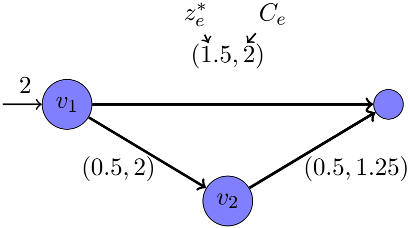

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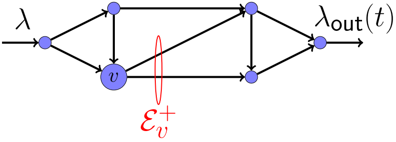

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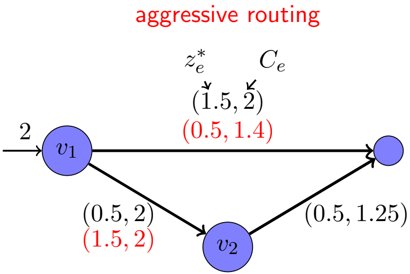

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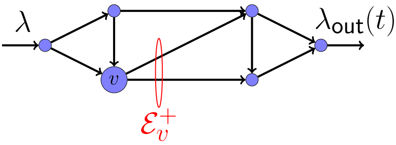

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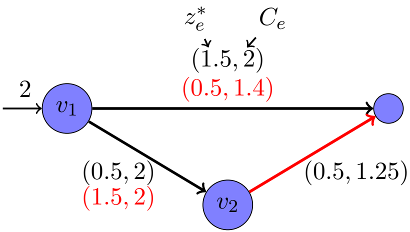
aggressive routing



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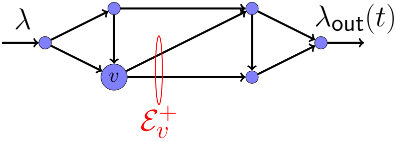

downstream cascades

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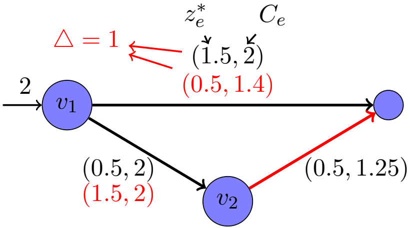
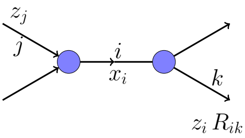


General Dynamical Network Flow



Mass Conservation

$$\dot{x} = \lambda + R(x)^T z(x) - z(x)$$
$$= \lambda + \left(R^T(x) - I\right) z(x)$$

x_i : mass on link i

λ_i : external inflow into link i

$R(x)$: sub-stochastic routing matrix

$z_i(x)$: outflow from link i

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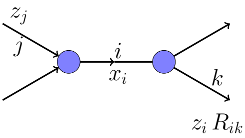

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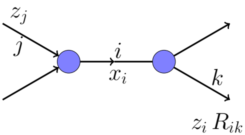

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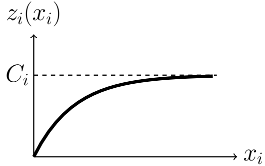
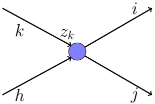
Objectives

- Stability
- Robustness
- Optimal control

Stability of Simple Nonlinear Model





- $R_{ki} = R_{hi}$; $R_{ii} = 0$
- $\frac{\partial}{\partial x_j} R_{ki}(x) \geq 0 \quad i \neq j$
- Decentralized control

$$\dot{x} = \lambda + \left(R^T(x) - I\right) z(x) =: g(x)$$

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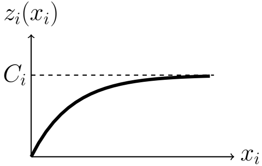

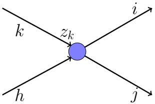

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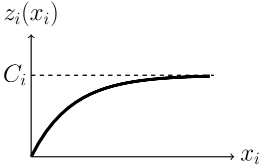

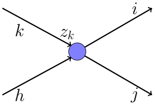

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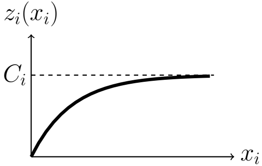
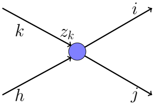
≥ 0

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$\sum \leq 0$

Stability of Simple Nonlinear Model





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$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \quad \geq 0$$

$\sum \leq 0$

ℓ_1 Contraction Principle

If $[\partial g_i / x_j]_{ij}$ is compartmental $\forall\, x$,

then $\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{x}(t) - \hat{x}(t)\|_1 \leq 0 \quad \forall\, \tilde{x}(0),\, \hat{x}(0)$

State-dependent Queue: Human-in-the-loop Systems

$$\dot{x} = \frac{b - x}{\tau}, \qquad \tau > 0$$

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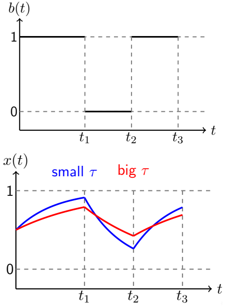

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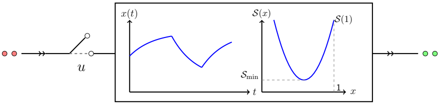

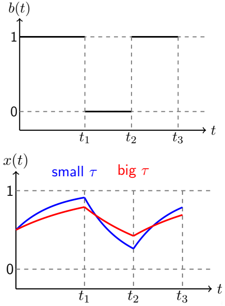

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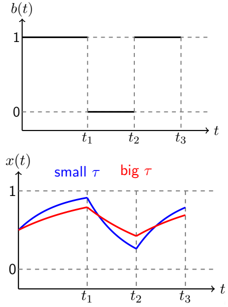
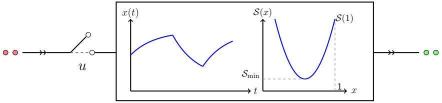
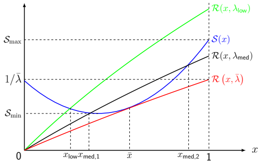
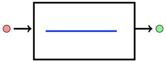
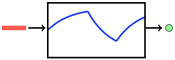
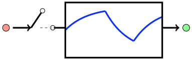






- Threshold policy is throughput optimal:
$$u_{\mathrm{T}}(t) = \begin{cases} \mathrm{ON} & \text{if } x(t) \leq \bar{x}, \\ \mathrm{OFF} & \text{otherwise.} \end{cases}$$

State-Dependent Queue: Horizontal Traffic Queue

			
stability	$\sigma(\text{service}) > 0$ $=$ $\sigma(\text{service}) = 0$	N/A	$\sigma(\text{service}) > 0$ $>$ $\sigma(\text{service}) = 0$

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	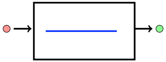	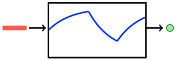	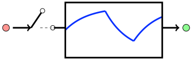
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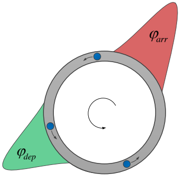
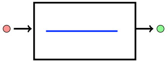
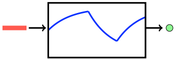
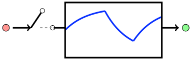


service rate $= \sum_i (x_{i+1} - x_i)^m$

State-Dependent Queue: Horizontal Traffic Queue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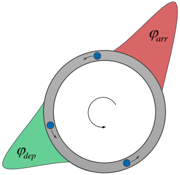

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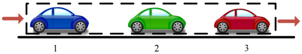
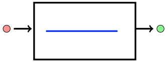
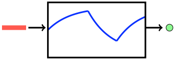
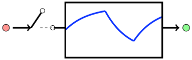


State-Dependent Queue: Horizontal Traffic Queue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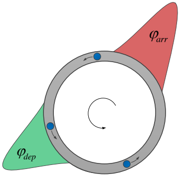

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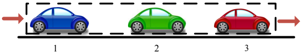
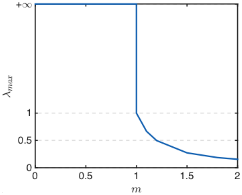
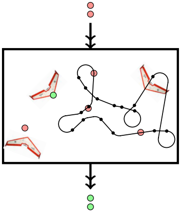




TSP under Differential Constraints

$$\dot{n} = \lambda - \frac{n}{\mathsf{TSP}(n)}$$

- Euclidean: $\mathsf{TSP}(n) = \Theta(n^{1/2})$
- Dubins, Diff Drive, ...
  $\mathsf{TSP}(n) = \Theta(n^{2/3})$



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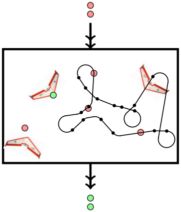
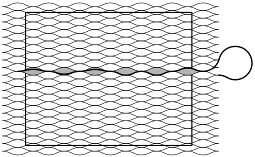
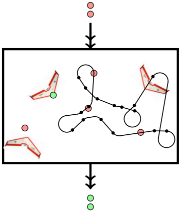




Phase 1

$$\dot{n} = \lambda - \frac{n}{\mathsf{TSP}(n)}$$



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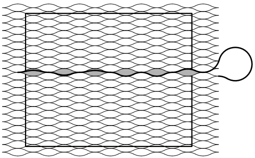
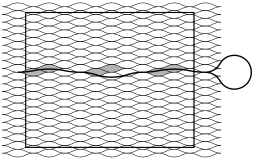


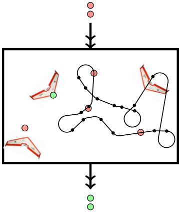
Phase 1



Phase 2

TSP under Differential Constraints

$$\dot{n} = \lambda - \frac{n}{\mathsf{TSP}(n)}$$



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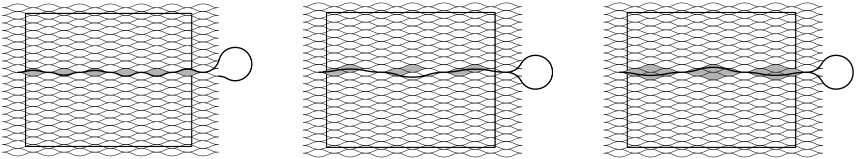
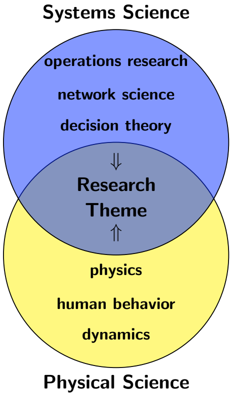


Phase 1 Phase 2 Phase 3

Current and Future Work



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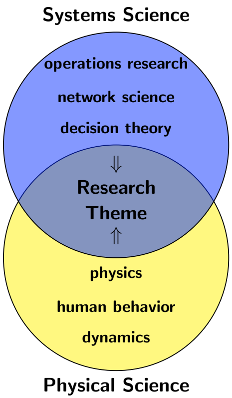
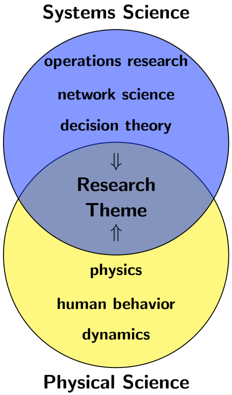


Current and Future Work

- {Cascades over Networks} + {Physics-based Failure and Recovery}

Current and Future Work



Current and Future Work

- {Cascades over Networks} + {Physics-based Failure and Recovery}
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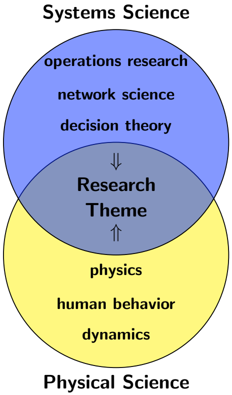

Current and Future Work

- {Cascades over Networks} + {Physics-based Failure and Recovery}
- {Game Theory} + {Geometric Strategy Space}
- {Compressed Sensing} + {Constrained Measurement Matrix}

Emphasis on fundamental performance limits